DEVELOPMENT OF FIFTH-ORDER FIVE-STAGE EXPLICIT ALMOST RUNGE-KUTTA METHODS

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Abstract
In this paper, two new Almost Runge-Kutta methods of orders five with five stages are introduced through a judicious and careful choice of free parameters to the general form of deriving order five methods. Appropriate convergence analysis established that the methods are consistent and stable, hence their convergence. The proposed methods, christened ARK5a and ARK5b, were implemented using sample initial value problems and the results compared with those of some existing ARK methods of the same order. Experimental results revealed that the methods are more efficient and effective than the existing methods by producing lesser errors and lesser computational rigour.

Keywords: Stability; convergence almost runge-kutta; order conditions

Introduction

The class of numerical methods known as the General Linear Methods came into being in 1996 as a result of the work of Butcher (1996), in an effort to formulate a general class of methods that incorporate the multivalue attributes of linear multistep methods into the multistage attributes of Runge-Kutta methods (Rattenbury, 2005). Almost Runge-Kutta (ARK) methods, introduced by Butcher (1997), are a special class of General Linear Methods whose properties have close affinity to those of explicit Runge-Kutta methods. ARK methods are able to attain the multivalue character of Linear Multistep Methods by allowing more than one value to be passed from step to step, even as the multistage nature of Runge-Kutta methods is retained. They share a lot of features with single step schemes and possess stability features of Runge-Kutta schemes.

Since the introduction of this class of methods, many researchers have contributed in no small measure towards improvement of the method for greater effectiveness and efficiency. In this direction, Rattenbury (2005) derived some Almost Runge-Kutta methods. This research which focused mainly on explicit schemes concentrated more on a unique fourth order numerical scheme for numerical integration of non-stiff differential equations, which when put into effect in the right way, operates like a fifth order scheme. She further derived low order diagonally implicit schemes for solving stiff differential equations. In Abraham (2010) some new ARK methods for the solution of non-stiff differential equations were derived; a comparison of relative performance of the methods with Runge-Kutta methods established that the ARK methods compete favourably with traditional Runge-Kutta methods. Alimi (2014) found out that the investigated effectiveness of Richardson Extrapolation Technique in estimating the error associated with Almost Runge-Kutta schemes. The outcome of the investigation indicates that Richardson extrapolation technique is not effective in achieving error estimates associated with Almost Runge-Kutta schemes. He also generated a numerical solver for carrying out experimentin numerical analysis. Furthermore, Ndanusa and Audu (2016a) derived two explicit ARK methods, a three stage third order method, and a four-stage third-order method for the numerical integration of initial value problems. In Ndanusa and Audu (2016b), two ARK methods of orders four and five (ARK5) were constructed. Numerical examples justified the introduction of these methods in that the methods performed better than some existing ARK methods of equal standing.

This research is aimed at developing some fifth order five stage Almost Runge-Kutta methods for numerical integration of initial value problems of ordinary differential equations.

Materials and Methods

Following Butcher (2008), the general fifth-order five-stage ARK method takes form

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 & c_1 & \frac{1}{2}c_1^2 \\ a_{21} & 0 & 0 & 0 & 0 & 1 & c_2 - a_{21} & \frac{1}{2}c_2^2 - a_{21}c_1 \\ a_{31} & a_{32} & 0 & 0 & 0 & 1 & c_3 - a_{31} - a_{32} & \frac{1}{2}c_3^2 - a_{31}c_1 - a_{32}c_2 \\ a_{41} & a_{42} & a_{43} & 0 & 0 & 1 & c_4 - a_{41} - a_{42} - a_{43} & \frac{1}{2}c_4^2 - a_{41}c_1 - a_{42}c_2 - a_{43}c_3 \\ b_1 & b_2 & b_3 & b_4 & 0 & 1 & b_0 & 0 \\ \hline b_1 & b_2 & b_3 & b_4 & 0 & 1 & b_0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & 0 & \beta_0 & 0 \end{array}\right] \tag{1}$$

where $c_5 = 1$; the Order conditions are:

$$\left.\begin{array}{c} b^T c = \frac{1}{2}, b^T c^2 = \frac{1}{3}, b^T c^3 = \frac{1}{4}, b^T c^4 = \frac{1}{5} \\ b_0 + b^T e = 1, b^T A = b^T(1 - C), \\ b^T(1-c)Ac = \frac{1}{24}, b^T(1-c)Ac^2 = \frac{1}{60}, \\ c_1 = -2\frac{exp_5(-\beta_5)}{\beta_5 exp_4(-\beta_5)}, \beta^T e + \beta_0 = 0, \\ \beta^T(I + \beta_5 A) = \beta_5 e_5^T \\ \left(1 + \frac{1}{2}\beta_5 c_1\right) b^T A^3 c = \frac{1}{120} \end{array}\right\} \tag{2}$$

From the order conditions (2) we obtain

$$c_1 = \frac{-2\left(1 - \beta_5 + \frac{1}{2}\beta_5^2 - \frac{1}{6}\beta_5^3 + \frac{1}{24}\beta_5^4 + \frac{1}{120}\beta_5^5\right)}{\beta_5\left(1 - \beta_5 + \frac{1}{2}\beta_5^2 - \frac{1}{6}\beta_5^3 + \frac{1}{24}\beta_5^4\right)} \tag{3}$$

$$a_{32} = \frac{2 - 5c_1}{120b_3(c_2^2 - c_1c_2 - c_2^2c_3 + c_1c_2c_3)} \tag{4}$$

$$a_{31} = \frac{\frac{1}{24} - b_2 a_{21} c_1(1 - c_2) - b_3 a_{32} c_2(1 - c_3)}{b_3 c_1(1 - c_3)} \quad (5)$$

$$a_{21} = \frac{2c_2(c_2 - c_1)}{c_1(2 - 5c_1)(2 + c_1\beta_5)} \quad (6)$$

$$a_{41} = \frac{b_1 - b_1 c_1 - b_2 a_{32} - b_3 a_{31}}{b_4} \quad (7)$$

$$a_{42} = \frac{b_1 - b_2 c_2 - b_3 a_{32}}{b_4} \quad (8)$$

$$a_{43} = \frac{b_3 - b_3 c_3}{b_3} \quad (9)$$

$$\left.\begin{array}{c} b_1 c_1 + b_2 c_2 + b_3 c_3 + b_4 c_4 = \frac{1}{2} \\ b_1 c_1 + b_2 c_2 + b_3 c_3 + b_4 c_4 = \frac{1}{2} \\ b_1 c_1^3 + b_2 c_2^3 + b_3 c_3^3 + b_4 c_4^3 = \frac{1}{4} \\ b_1 c_1^4 + b_2 c_2^4 + b_3 c_3^4 + b_4 c_4^4 = \frac{1}{5} \\ b_0 + b_1 + b_2 + b_3 + b_4 = 1 \end{array}\right\} \quad (10)$$

Solving system (10) results in

$$b_1 = \frac{5c_2 + 5c_3 - 10c_2c_3 - 3}{60\,(c_1 - 1)c_1(c_1 - c_2)(c_1 - c_3)} \quad (11)$$

$$b_2 = \frac{10\,c_1c_3 - 5c_1 - 5c_3 + 3}{60c_2(c_1 - c_2)(c_2 - 1)(c_2 - c_3)} \quad (12)$$

$$b_3 = -\frac{1}{60}\left[\frac{10\,c_1c_2 - 5c_1 - 5c_2 + 3}{c_3(c_3 - 1)(c_2 - c_3)(c_1 - c_3)}\right] \quad (13)$$

$$b_4 = \frac{30c_1c_2c_3 - 20c_1c_2 - 20c_1c_3 - 20c_2c_3 + 15c_1 + 15c_2 + 15c_3 - c_2}{60(c_3 - 1)(c_2 - 1)(c_1 - 1)} \quad (14)$$

$$b_0 = \frac{30c_1c_2c_3 - 10c_1c_2 - 10c_1c_3 - 10c_2c_3 + 5c_1 + 5c_2 + 5c_3 - 3}{60\,c_1c_2c_3} \quad (15)$$

And from $\beta^T(I + \beta_5 A) = \beta_5 e_5^T$ of (2) results

$$\beta_1 + a_{21}a_{32}a_{43}b_4\beta_5^5 - a_{21}a_{32}b_3\beta_5^4 - a_{21}a_{42}b_4\beta_5^4 + a_{21}b_2\beta_5^4 - a_{31}a_{43}b_4\beta_5^4$$
$$+a_{31}b_3\beta_5^3 + a_{41}b_4\beta_5^3 - b_1\beta_5^2, \beta_2 - a_{32}a_{43}b_4\beta_5^4 + a_{42}b_4\beta_5^3 + a_{32}b_3\beta_5^3$$
$$-b_2\beta_5^2, \beta_3 + a_{43}b_4\beta_5^3 - b_3\beta_5^2, \beta_4 - b_4\beta_5^2, \beta_5 = 0,0,0,0,\beta_5 \quad (16)$$

Thus,

$$\beta_1 = a_{21}a_{32}a_{43}b_4\beta_5^5 - a_{21}a_{32}b_3\beta_5^4 - a_{21}a_{42}b_4\beta_5^4$$
$$+a_{21}b_2\beta_5^4 - a_{31}a_{43}b_4\beta_5^4 + a_{31}b_3\beta_5^3 + a_{41}b_4\beta_5^3 - b_1\beta_5^2 \quad (17)$$

$$\beta_2 = -a_{32}a_{43}b_4\beta_5^4 + a_{42}b_4\beta_5^3 + a_{32}b_3\beta_5^3 - b_2\beta_5^2 \quad (18)$$

$$\beta_3 = a_{43}b_4\beta_5^3 - b_3\beta_5^2 \quad (19)$$

$$\beta_4 = -b_4\beta_5^2 \quad (20)$$

$$\beta_5 = \beta_5 \quad (21)$$

$$\beta_0 = -\beta_1 - \beta_2 - \beta_3 - \beta_4 - \beta_5 \quad (22)$$

And the following fifth-order five-stage ARK schemes, christened ARK5a and ARK5b are obtained

ARK5a with $c^T = \left[\frac{53}{150}, \frac{1}{2},, \frac{3}{4}, 1,1\right]$

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 & \frac{53}{150} & \frac{2809}{45000} \\ \frac{12375}{23744} & 0 & 0 & 0 & 0 & 1 & -\frac{503}{23744} & -\frac{53}{896} \\ \frac{95625}{74624} & \frac{833}{3520} & 0 & 0 & 0 & 1 & -\frac{26053}{33920} & -\frac{371}{1280} \\ -\frac{982125}{1466828} & -\frac{1455}{407} & \frac{7760}{4403} & 0 & 0 & 1 & \frac{191193}{54908} & \frac{2491}{2072} \\ \frac{4218750}{6729569} & -\frac{8}{33} & \frac{160}{357} & \frac{37}{582} & 0 & 1 & \frac{11}{106} & 0 \\ \hline \frac{4218750}{6729569} & -\frac{8}{33} & \frac{160}{357} & \frac{37}{582} & 0 & 1 & \frac{11}{106} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{48750}{5141} & -\frac{32}{3} & 0 & -\frac{296}{291} & 4 & 0 & -\frac{286}{159} & 0 \end{array}\right] \quad (23)$$

ARK5b with $c^T = \left[\frac{53}{150}, \frac{1}{3}, \frac{2}{3}, 1,1\right]$

$$\left[\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 & \frac{53}{150} & \frac{2809}{45000} \\ -\frac{1125}{23744} & 0 & 0 & 0 & 0 & 1 & \frac{27119}{71232} & \frac{583}{8064} \\ \frac{6480125}{2386272} & -\frac{329}{201} & 0 & 0 & 0 & 1 & -\frac{983389}{2386272} & -\frac{51781}{270144} \\ -\frac{7763140375}{257788608} & \frac{6499}{231} & \frac{6499}{3619} & 0 & 0 & 1 & \frac{2165363}{1828288} & \frac{116971}{206976} \\ -\frac{625000}{241627} & \frac{23}{8} & \frac{201}{376} & \frac{77}{376} & 0 & 1 & \frac{33}{424} & 0 \\ \hline -\frac{625000}{241627} & \frac{23}{8} & \frac{201}{376} & \frac{77}{376} & 0 & 1 & \frac{33}{424} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{110286250}{724881} & \frac{454}{3} & \frac{134}{47} & -\frac{154}{97} & 4 & 0 & -\frac{236}{53} & 0 \end{array}\right] \quad (24)$$

Convergence Analysis

For the scheme ARK5a, its V matrix is

$$V = \begin{bmatrix} 1 & \frac{11}{106} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{286}{159} & 0 \end{bmatrix} \quad (25)$$

From the characteristic polynomial of V

$$\rho(\lambda) = \det(\lambda \mathrm{I}_3 - V) \tag{26}$$

$$\begin{vmatrix} \lambda - 1 & -\frac{11}{106} & 0 \\ 0 & \lambda & 0 \\ 0 & \frac{286}{159} & \lambda \end{vmatrix} = \lambda^3 - \lambda^2 \tag{27}$$

Following Cayley-Hamilton theorem,

$$\rho(V) = V^3 - V^2 = 0 \tag{28}$$

$$\begin{bmatrix} 1 & \frac{11}{106} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & \frac{11}{106} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29}$$

It implies

$$V^3 = V^2 \tag{30}$$

Similarly, $V^4 - V^2 = 0 \;\; => V^4 = V^2, \cdots, \text{and } V^n = V^2$ for every n greater than 2, meaning V^n is bounded which shows that the method is stable. Also, the method is consistent since it is of order $p = 5 > 1$. Therefore, ARK5a method is convergent.

And for ARK5b, the matrix is given as:

$$V = \begin{bmatrix} 1 & \frac{33}{424} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{236}{53} & 0 \end{bmatrix} \tag{31}$$

whose characteristic polynomial is

$$\rho(\lambda) = \lambda^3 - \lambda^2 \tag{32}$$

with the eigenvalues calculated to be $\lambda = 1, 0, 0$, and it satisfies the Cayley-Hamilton theorem thus

$$\rho(V) = V^3 - V^2 = 0 \tag{33}$$

$$\begin{bmatrix} 1 & \frac{33}{424} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & \frac{33}{424} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{34}$$

Consequently,

$$V^4 - V^2 = 0 \quad => V^4 = V^2 \cdots \; V^n = V^2 \tag{35}$$

for every n greater than 2, meaning V^n is bounded; thus the method is stable. The method is also consistent since it it of order $p = 5 > 1$. Therefore the scheme ARK5b is convergent.

Numerical Experiments

The proposed schemes ARK5a AND ARK5b are applied to problems 1 and 2 and the results compared with those of Ndanusa and Audu (2016b) in order to ascertain their efficacy. It is instructive to note that the fifth order method of Ndanusa and Audu (2016b) is more accurate than the methods of Abraham (2010) and Alimi (2014). The results are presented in Tables 1 and 2.

Problem 1:

$$\left.\begin{array}{l} y' = x + y, \quad y(0) = 1 \\ h = 0.1, \quad x \in [0,1] \\ y(x) = 2e^{x} - x - 1 \end{array}\right\} \qquad (36)$$

Problem 2

$$\left.\begin{array}{l} y' = x + 2y, \quad y(0) = 1 \\ h = 0.1, \quad x \in [0,1] \\ y(x) = -\frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{2x} \end{array}\right\} \qquad (37)$$

Table 1: Results of Problem 1

x	y_{exact}	$y_{(*)}$	$Er_{(*)}$	$y_{(ARK5a)}$	$Er_{(ARK5a)}$	$y_{(ARK5b)}$	$Er_{(ARK5b)}$
0.0	1.000000000	1.000000000	0.000000000	1.000000000	0.000000000	1.000000000	0.000000000
0.1	1.110341836	1.110341835	0.000000001	1.110341836	0. 000000000	1.110341835	0. 00000001
0.2	1.242805516	1.242805512	0.000000004	1.242805513	0.000000003	1. 242805513	0.000000003
0.3	1.399717615	1.399717607	0.000000008	1.399717608	0.000000007	1. 399717609	0.000000006
0.4	1.583649395	1.583649382	0.000000012	1.583649385	0.000000010	1. 583649384	0.000000011
0.5	1.797442541	1.797442523	0.000000018	1.797442526	0.000000015	1. 797442526	0.000000016
0.6	2.044237601	2.044237576	0.000000025	2.044237580	0.000000021	2. 044237579	0.000000022
0.7	2.327505415	2.327505382	0.000000033	2.377505387	0.000000028	2. 377505386	0.000000029
0.8	2.651081857	2.651081815	0.000000042	2.651081820	0.000000037	2. 651081819	0.000000038
0.9	3.019206222	3.019206169	0.000000052	3.019206175	0.000000047	3. 019206174	0.000000048
1.0	3.436563657	3.436563592	0.000000065	3.436563598	0.000000059	3. 436563597	0.000000060

Table 2: Results of Problem 2

x	y_{exact}	$y_{(*)}$	$Er_{(*)}$	$y_{(ARK5a)}$	$Er_{(ARK5a)}$	$y_{(ARK5b)}$	$Er_{(ARK5b)}$
0.0	1.00000000	1.00000000	0.00000000	1.00000000	0.00000000	1.00000000	0.00000000
0.1	1.22675344	1.22675338	0.00000006	1.22675340	0.00000004	1.22675340	0.00000004
0.2	1.51478087	1.51478065	0.00000021	1.51478067	0.00000019	1.51478067	0.00000019
0.3	1.87764850	1.87764806	0.00000043	1.87764809	0.00000040	1.87764809	0.00000041
0.4	2.33192616	2.33192541	0.00000074	2.33192545	0.00000112	2.33192545	0.00000112
0.5	2.89785228	2.89785112	0.00000116	2.89785116	0.00000168	2.89785116	0.00000168
0.6	3.60014615	3.60014442	0.00000173	3.60014447	0.00000243	3.60014447	0.00000243
0.7	4.46899995	4.46899746	0.00000249	4.46899752	0.00000343	4.46899752	0.00000343
0.8	5.54129053	5.54128702	0.00000350	5.54128709	0.00000476	5.54128709	0.00000476
0.9	6.86205933	6.86205447	0.00000485	6.86205456	0.00000476	6.86205456	0.00000476
1.0	8.48632012	8.48631350	0.00000662	8.48631361	0.00000651	8.48631361	0.00000651

Discussion of Results

Tables 1 and 2 depict the results of applying the proposed methods ARK5a and ARK5b to the test Problems 1 and 2. From the tables, x stands for the integration points at which the solutions are sought, y_{exact} is the exact solutions at the various integration points, $y_{(*)}$ is the approximate solutions for the method of Ndanusa and Audu (2016b), $\boldsymbol{y}_{(\boldsymbol{ARK5a})}$ is the approximate solution of ARK5a, $\boldsymbol{y}_{(\boldsymbol{ARK5b})}$ is the approximate solution of ARK5b, while $\boldsymbol{Er}_{(*)}$, $\boldsymbol{Er}_{(\boldsymbol{ARK5a})}$ and $\boldsymbol{Er}_{(\boldsymbol{ARK5b})}$ are the errors produced by the methods of Ndanusa and Audu (2016b), ARK5a and ARK5b respectively. In summary, it is evident that the proposed methods ARK5a and ARK5b exhibit lesser errors than the method of Ndanusa and Audu (2016b), and by extension the methods of Abraham (2010) and Alimi (2014).

Conclusion

This research has succeeded in producing two new explicit Almost Runge-Kutta methods each of order 5. The methods are proven to be convergent by subjecting them to standard convergence criteria. Numerical examples further established that the methods are not only effective but also compare favourably with other known methods in existence.

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